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# The M5-brane Hamiltonian

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## Abstract

We obtain the Hamiltonian form of the world-volume action for the M5-brane in a general  $D = 11$  supergravity background. We use this result to obtain a new version of the covariant M5-brane Lagrangian in which the tension appears as a dynamical variable, although this Lagrangian has some unsatisfactory features which we trace to peculiarities of the null limit. We also show that the M5-brane action is invariant under all (super)isometries of the background. © 1998 Elsevier Science B.V.

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## 1. Introduction

The essential ‘ingredients’ of M-theory that are additional to those of 11-dimensional ( $D = 11$ ) supergravity are the supermembrane, or M2-brane, and the M5-brane. The world-volume action for the supermembrane has been known for more than ten years [1]. In contrast, the full world-volume action for the M5-brane has been known for only a year or so and its implications are still being explored. The M5-brane action is essentially one for an interacting six-dimensional  $(2, 0)$  supersymmetric gauge theory based on the  $(2, 0)$  antisymmetric tensor supermultiplet [2]. The self-duality of the 3-form field strength of this supermultiplet presents serious obstacles to the construction of a

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six-dimensional Lorentz covariant action, some of which are inevitable at the quantum level [3]. This is not a problem at the level of field equations, however, although the self-duality constraint involves non-linearities that would be hard to guess [4]. The full field equations were found in superfield form in [5]. A covariant component action involving an additional scalar gauge field was presented in [6], although it is restricted to backgrounds admitting a nowhere-null vector field. This action was used in [7] to determine the central charge structure of the M5-brane supertranslation algebra in a vacuum background. Alternatively, an action can be constructed by relaxing the requirement of six-dimensional Lorentz covariance to five-dimensional Lorentz covariance [8]. These various formulations of the M5-brane action are now known to be equivalent [9,10].

In this paper we shall present another formulation of the M5-brane action: the Hamiltonian formulation. The Hamiltonian formulation of the M2-brane can be found in [11]. The space/time split implicit in the Hamiltonian formulation has the advantage that the self-duality constraint is no problem and, in fact, is reduced to a simple *linear* constraint on phase space. Also, the Hamiltonian formulation is the natural one for investigations of static solutions that minimise the energy, and some of the results obtained here were advertised and then used for this purpose in [12].

The passage to the Hamiltonian formulation from the Lagrangian one is simplest if one starts from the covariant action of [6] because much of the required space/time split can be achieved by the choice of temporal gauge for the auxiliary scalar gauge field of this action (the ‘PST’ field). We shall therefore begin by reviewing those elements of this formulation that are essential to the subsequent steps. In doing so we take the opportunity to show, following a similar recent demonstration for super D-branes [13], that the M5-brane action is invariant under all (super)isometries of the  $D = 11$  supergravity background, provided that the 2-form gauge potential is assigned an appropriate transformation. This observation acquires importance in light of the recent construction of an interacting conformal invariant antisymmetric tensor field theory via gauge fixing of the bosonic sector of the M5-brane action in an  $\text{adS}_7 \times S^4$  background [14]. Specifically, it implies that the full M5-brane action in this background is invariant under the full  $OSp(6, 2|4)$  isometry supergroup of the  $\text{adS}_7 \times S^4$  solution [15] of  $D = 11$  supergravity. This symmetry will be realized ‘on the brane’ as a non-linearly realized six-dimensional superconformal invariance.

For all branes other than the M5-brane it is known that the tension can be replaced by a dynamical  $p$ -form world-volume gauge potential, leading to a Lagrangian that is strictly invariant under background isometries (as against invariant up to the addition of a total derivative) by virtue of appropriate transformations of the new world-volume field [16,13]. It was suggested in [17] that this may not be possible for the M5-brane. On the other hand, a version of the M5-brane action with dynamical tension was found in [18], although it did not incorporate the self-duality constraint (which had to be imposed separately). Thus, at present, the status of dynamical tension in the M5-brane case is unclear and one of the aims of this paper is to shed some light on this point. The Hamiltonian formulation provides the means to do so; in this formulation it is a simple matter to elevate the tension to the status of a dynamical variable. One can then pass to

the corresponding covariant Lagrangian formulation in which the tension is replaced by a 5-form gauge potential. We find that this Lagrangian has some unsatisfactory features, which we trace to peculiarities of the tensionless limit.

## 2. The M5-brane action and its rigid symmetries

Any solution of the field equations of  $D = 11$  supergravity is a consistent background for the M5-brane. These backgrounds can be presented as tensors on  $D = 11$  superspace, which is parameterised by the coordinates  $Z^M = (X^m, \theta^\alpha)$ . Specifically, we need a  $D = 11$  supervielbein  $E_M^A$ , which are the coordinate basis components of the frame 1-forms  $E^A = (E^a, E^\alpha)$ , where  $E^a$  and  $E^\alpha$  are, respectively, a vector and a Majorana spinor of the  $D = 11$  Lorentz group. We also need a superspace 3-form gauge potential  $C^{(3)}$  and a 6-form gauge potential  $C^{(6)}$ . Their gauge transformations are

$$\begin{aligned}\delta C^{(3)} &= d\Lambda^{(2)}, \\ \delta C^{(6)} &= d\Lambda^{(5)} - \frac{1}{2}\Lambda^{(2)}R^{(4)},\end{aligned}\quad (1)$$

where  $\Lambda^{(2)}$  and  $\Lambda^{(5)}$  are 2-form and 5-form parameters, respectively, and their gauge-invariant field strengths are

$$\begin{aligned}R^{(4)} &= dC^{(3)}, \\ R^{(7)} &= dC^{(6)} + \frac{1}{2}C^{(3)}R^{(4)},\end{aligned}\quad (2)$$

where the exterior product of forms is understood. The relative sign in the definition of  $R^{(7)}$  differs from, e.g. Ref. [7], because we use here the convention that the exterior superspace derivative  $d$  ‘acts from the right’, i.e.

$$d(PQ) = P dQ + (-1)^q (dP)Q \quad (3)$$

for  $p$ -form  $P$  and  $q$ -form  $Q$ . The on-shell superfield constraints of  $D = 11$  supergravity imply, *inter alia*, that the bosonic component of  $R^{(7)}$  is the  $D = 11$  Hodge dual of the bosonic component of  $R^{(4)}$ .

Before turning to the M5-brane itself, let us note that a Killing vector superfield  $\xi(Z)$  is one for which

$$\mathcal{L}_\xi(E^a \otimes_s E^b) \eta_{ab} = 0, \quad (4)$$

where  $\mathcal{L}_\xi$  denotes the Lie derivative with respect to  $\xi$ , and  $\eta$  is the  $D = 11$  Minkowski metric. This is the superfield version of the Killing condition. By an ‘isometry’ of the supergravity background we shall mean a transformation generated by a Killing vector field for which, additionally,

$$\mathcal{L}_\xi R^{(4)} = 0, \quad \mathcal{L}_\xi R^{(7)} = 0. \quad (5)$$

It is convenient to summarize the action of the complete set of such Killing vector superfields  $\xi_\alpha$  by means of a BRST operator  $s$ , so that

$$sZ^M = c^\alpha \xi_\alpha^M \equiv c^M, \quad sC^\alpha = \frac{1}{2} c^\gamma c^\beta f_{\beta\gamma}{}^\alpha, \quad (6)$$

where  $c^\alpha$  is a set of *constant* BRST ‘fields’ and  $f_{\beta\gamma}{}^\alpha$  are the structure constants of the Lie algebra of Killing vector fields. It follows that  $s^2 c^\alpha \equiv 0$ , and that  $s^2 Z^M \equiv 0$  (and hence that the action of  $s^2$  on any superfield vanishes identically). Thus, Eq. (5) reduces to  $sR^{(4)} = sR^{(7)} = 0$ , which implies that<sup>3</sup>

$$sC^{(3)} = d\Delta_2, \quad sC^{(6)} = d\Delta_5 - \frac{1}{2}\Delta_2 R^{(4)}, \quad (7)$$

where  $\Delta_p$  is a ghost-valued superspace  $p$ -form.

Now, let  $\sigma^i$  ( $i = 0, 1, \dots, 5$ ) be the world-volume coordinates of the 5-brane and  $f$  a map from the world-volume to superspace. We take the world-volume metric to be the pullback  $g = f^*(E^a \otimes_s E^b) \eta_{ab}$ . It is manifestly invariant under isometries of the background. The world-volume fields include, in addition to  $Z^M(\sigma)$ , a 2-form world-volume gauge potential  $A(\sigma)$  with ‘modified’ 3-form field strength<sup>4</sup>

$$H = dA - C^{(3)}, \quad (8)$$

where  $C^{(3)}$  is now to be understood as the pullback of the corresponding superspace 3-form gauge potential. Clearly,  $H$  is invariant under isometries of the background provided that we choose

$$sA = \Delta_2. \quad (9)$$

We shall also need to define<sup>5</sup>

$$\tilde{H}^{ij} = \frac{1}{6\sqrt{g}g^{lm}\partial_l a \partial_m a} \epsilon^{ijklj'k'} (\partial_k a) H_{i'j'k'}, \quad (10)$$

where  $g = \det g_{ij}$  and  $a(\sigma)$  is the ‘PST’ scalar gauge field. The 2-form  $\tilde{H}$  is also invariant under isometries of the background if we take  $sa = 0$ . Having made the above definitions, and introducing the 5-brane tension  $T$ , we may write down the M5-brane Lagrangian of [6]. This was originally written in the form

$$L_{M5} = T[L_{DBI} + \tilde{H}H + L_{WZ}], \quad (11)$$

where

$$L_{DBI} = -\sqrt{-\det(g_{ij} + \tilde{H}_{ij})}, \quad (12)$$

is a type of Dirac–Born–Infeld action, and

$$\tilde{H}H = \frac{1}{24(\partial a)^2} (\partial_i a) \epsilon^{ijklj'k'} H_{i'j'k'} H_{jkl} g^{ll'} \partial_{l'} a. \quad (13)$$

The last, Wess–Zumino (WZ), term will be given below.

<sup>3</sup> The same symbols were used in Ref. [7] with a related but not identical meaning. Here we follow the notation of Ref. [13].

<sup>4</sup> Here we adopt a normalization of the two-form potential  $A$  that differs by a factor of three from Ref. [7].

<sup>5</sup> Our metric signature is ‘mostly plus’.

Here we shall start from the equivalent Lagrangian

$$L = L_0 + TL_{\text{WZ}}, \quad (14)$$

where

$$L_0 = \frac{1}{2v} \det(g_{ij} + \tilde{H}_{ij}) - T^2 v + T\tilde{H}H \quad (15)$$

with  $v(\sigma)$  an independent world-volume scalar density. If we take  $sv = 0$ , in addition to the previously assigned BRST transformations, then  $L_0$  is clearly invariant under isometries of the background. The WZ term is

$$\begin{aligned} L_{\text{WZ}} &= \frac{1}{6!} \epsilon^{ijklmn} [C_{ijklmn}^{(6)} + 10H_{ijk}C_{lmn}^{(3)}] \\ &= \star(C^{(6)} + \frac{1}{2}HC^{(3)}), \end{aligned} \quad (16)$$

where  $\star$  is the world-volume Hodge dual, and  $C^{(6)}$  and  $C^{(3)}$  are now understood to be the pullbacks to the world-volume of the corresponding superspace forms. Now,

$$s(C^{(6)} + \frac{1}{2}HC^{(3)}) = d(\Delta_5 + \frac{1}{2}H\Delta_2), \quad (17)$$

where the ghost-valued forms  $\Delta_p$  are also to be understood now as pullbacks to the world-volume of the corresponding superspace  $p$ -forms (recall that  $d$  ‘acts from the right’).

Thus,  $L_{\text{WZ}}$  is invariant under isometries of the background up to a possible total derivative. This is sufficient to ensure the existence of a conserved world-volume current associated with each isometry of the background, although the total derivative term, and  $\Delta_2$  if non-zero, can lead to central terms in the algebra of isometries, as discussed in [7] for the case of a vacuum  $D = 11$  background.

### 3. The Hamiltonian formulation

As shown in [6], the scalar field  $a$  is subject to a gauge transformation that allows the choice of ‘temporal gauge’  $a = \sigma^0 \equiv t$ . This choice breaks the  $SO(1,5)$  Lorentz group to the  $SO(5)$  rotation group, but this is in any case an expected feature of the Hamiltonian formulation. Our goal is to separate in the Lagrangian  $L$  all terms involving time derivatives of the world-volume fields. Let us set  $\sigma^i = (t, \sigma^{\underline{a}})$ , ( $\underline{a} = 1, \dots, 5$ ). Then, in the  $a = t$  gauge we have

$$\tilde{H}H = \frac{1}{24} \epsilon^{\underline{abcde}} H_{\underline{cde}} H_{\underline{ab0}} - V_{\underline{a}}^5 g^{\underline{ab}} g_{\underline{b0}}, \quad (18)$$

where  ${}^5g^{\underline{ab}}$  is the inverse of  $g_{\underline{ab}}$  (rather than the space/space components of  $g^{ij}$ ) and

$$V_{\underline{f}} = \frac{1}{24} \epsilon^{\underline{abcde}} H_{\underline{cde}} H_{\underline{abf}}. \quad (19)$$

In addition, the only non-vanishing component of  $\tilde{H}^{ij}$  in the gauge  $a = t$  is

$$\tilde{H}^{ab} = \frac{1}{6\sqrt{5}g} \epsilon^{\underline{abcde}} H_{\underline{cde}}, \quad (20)$$

where  ${}^5g$  is the determinant of the world-space 5-metric  $g_{ab}$ . It follows that

$$\det(g_{ij} + \tilde{H}_{ij}) = (g_{00} - g_{0a} {}^5g^{ab} g_{0b}) \det^5(g + \tilde{H}), \quad (21)$$

where

$$\tilde{H}_{ab} = g_{ac} g_{bd} \tilde{H}^{cd}, \quad \det^5(g + \tilde{H}) = \det(g_{ab} + \tilde{H}_{ab}). \quad (22)$$

If we now define

$$\lambda = v / \det^5(g + \tilde{H}) \quad (23)$$

we find that

$$L_0 = \frac{1}{2\lambda} (g_{00} - g_{0a} {}^5g^{ab} g_{0b}) - TV^a g_{0b} - \frac{1}{2} \lambda T^2 \det^5(g + \tilde{H}) + \frac{T}{24} \epsilon^{abcde} H_{cde} H_{ab0}, \quad (24)$$

where  $V^a = {}^5g^{ab} V_b$ .

We must now make the space/time split in the WZ term. We have

$$L_{WZ} = \dot{Z}^M \left[ C_M - \frac{1}{24} \epsilon^{abcde} \dot{Z}^M C_{Mab}^{(3)} H_{cde} \right] + \frac{1}{24} \epsilon^{abcde} C_{cde}^{(3)} H_{ab0}. \quad (25)$$

Here we have introduced the world-space scalar

$$C_M = *i_M C^{(6)}, \quad (26)$$

where  $i_M$  indicates contraction with the vector field  $\partial/\partial Z^M$  (so that  $i_M C^{(6)}$  is a 5-form, which we restrict to the world-space) and  $*$  is the *world-space* Hodge dual.

We have now implicitly arrived at a space/time split form of the total M5-brane Lagrangian  $L = L_0 + TL_{WZ}$ . It will be convenient to write this result as

$$L = L_1 + L_2, \quad (27)$$

where  $L_1$  includes all but the last term (with the time component of  $H$ ) in (24) and  $L_2$  is the rest. An equivalent form for  $L_1$  is

$$L_1 = \tilde{P} \cdot \Pi_t - s^a (\tilde{P} \cdot \Pi_a + TV_a) - \frac{1}{2} \lambda \left[ (\tilde{P} + TV^a \Pi_a)^2 + T^2 \det^5(g + \tilde{H}) \right], \quad (28)$$

where  $\tilde{P}^a$  and  $s^a$  are new independent variables and we have set

$$(\Pi_i)^a \equiv E_i^a = \partial_i Z^M E_M^a. \quad (29)$$

Recall that the indices  $a, b, \dots$  denote  $D = 11$  Lorentz vectors, which are contracted with the Minkowski metric  $\eta$ . The equivalence may be established by successive elimination of  $\tilde{P}$  and  $s^a$ .

The Lagrangian  $L_2$  is

$$L_2 = T \dot{Z}^M \left[ C_M - \frac{1}{24} \epsilon^{abcde} C_{Mab}^{(3)} H_{cde} \right] + \frac{1}{24} T \epsilon^{abcde} H_{ab0} (dA)_{cde}. \quad (30)$$

Now,

$$H_{ab0} = \dot{A}_{ab} + 2\partial_{[a} A_{b]0} - \dot{Z}^M C_{Mab}^{(3)} \quad (31)$$

so, omitting a total derivative term, we have

$$L_2 = T\dot{Z}^M \hat{C}_M + \frac{1}{8} T \varepsilon^{abcde} \dot{A}_{ab} \partial_c A_{de}, \quad (32)$$

where

$$\begin{aligned} \hat{C}_M &= C_M - \frac{1}{24} \varepsilon^{abcde} C_{Mab}^{(3)} [C_{cde}^{(3)} + 2H_{cde}] \\ &= * [i_M C^{(6)} - \frac{1}{2} i_M C^{(3)} (C^{(3)} + 2H)]. \end{aligned} \quad (33)$$

We can further rewrite this as

$$L_2 = T\dot{Z}^M \hat{C}_M + \frac{1}{2} \Pi^{ab} \dot{A}_{ab}, \quad (34)$$

where

$$\Pi^{ab} = \frac{1}{4} T \varepsilon^{abcde} \partial_c A_{de}. \quad (35)$$

Putting all these results together we can write the M5-brane lagrangian in the form

$$L = \tilde{P} \cdot \Pi_t + \frac{1}{2} \Pi^{ab} \dot{A}_{ab} - \lambda \mathcal{H} - s^a \tilde{\mathcal{H}}_a + \sigma_{ab} \mathcal{K}^{ab} + T\dot{Z}^M \hat{C}_M, \quad (36)$$

where

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} [(\tilde{P} + TV^a \Pi_a)^2 + T^2 \det^5(g + \tilde{H})], \\ \tilde{\mathcal{H}}_a &= (\tilde{P} \cdot \Pi_a + TV_a), \\ \mathcal{K}^{ab} &= \Pi^{ab} - \frac{1}{4} T \varepsilon^{abcde} \partial_c A_{de}. \end{aligned} \quad (37)$$

We have now arrived at a ‘half-way house’ on the way to the fully canonical phase space form of the M5-brane Lagrangian, and it is convenient to pause here to assess the situation. Note, in particular, that the constraint  $\mathcal{K}^{ab} = 0$  can be written in differential form notation on world-space as

$$\Pi = \frac{1}{2} T * (dA), \quad (38)$$

which implies the Gauss law constraint  $d * \Pi = 0$ . In other words, the constraint  $\mathcal{K}^{ab} = 0$  ensures that the Bianchi identity for  $dA$  implies the Gauss law, as expected in a self-dual antisymmetric tensor field theory. Thus *the self-duality constraint that is so problematic in the Lagrangian reduces to a set of simple linear constraints on the phase space pair  $(\Pi, A)$* . However, these constraints are second class, in Dirac’s terminology, and this leads to problems upon quantization. One must either solve the constraints (which in general leads to non-locality) or convert them into first class ones by the addition of auxiliary variables. As explained in [19], the latter approach leads to the infinite-field formulation of chiral antisymmetric tensors of [20,21]. The reason that we find these bosonic second-class constraints is that we fixed the gauge for the PST field  $a$  in the passage to the Hamiltonian formulation. Had we not fixed this gauge invariance we would have found only first class bosonic constraints with an additional constraint associated with the additional gauge invariance, as found for the free chiral  $D = 6$  2-form field in [22].



From (36) we see that the momentum  $P_M$  conjugate to  $Z^M$  is given by

$$P_M = E_M^a \tilde{P}_a + T \hat{C}_M. \quad (39)$$

Solving for  $\tilde{P}$  we have

$$\tilde{P}_a = E_a^M P_M - T E_a^M \hat{C}_M. \quad (40)$$

The remaining information contained in (39) is the fermionic constraint

$$P_\mu - E_\mu^a E_a^M P_M = T [\hat{C}_\mu - E_\mu^a E_a^M \hat{C}_M], \quad (41)$$

which is equivalent to

$$E_\mu^a E_a^M (P_M - T \hat{C}_M) = 0. \quad (42)$$

Since  $E_\mu^a$  is invertible, this constraint is equivalent to  $S_\alpha = 0$ , where

$$S_\alpha = E_\alpha^M (P_M - T \hat{C}_M). \quad (43)$$

This constraint can be imposed by a new spinorial Lagrange multiplier  $\zeta^\alpha$ . It can also be used to simplify the constraint imposed by  $s^a$  since

$$\tilde{\mathcal{H}}_a = \mathcal{H}_a + \partial_a Z^M E_M^a S_\alpha, \quad (44)$$

where

$$\mathcal{H}_a = \partial_a Z^M P_M + T (V_a - \partial_a Z^M \hat{C}_M). \quad (45)$$

We thus arrive at the M5-brane Lagrangian in fully canonical form

$$L = \dot{Z}^M P_M + \frac{1}{2} \Pi^{ab} \dot{A}_{ab} - \lambda \mathcal{H} - s^a \mathcal{H}_a + \sigma_{ab} \mathcal{K}^{ab} - \zeta^\alpha S_\alpha, \quad (46)$$

where

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} [\mathcal{P}^2 + T^2 \det^5(g + \tilde{H})], \\ \mathcal{H}_a &= \partial_a Z^M P_M + T (V_a - \hat{C}_a), \\ \mathcal{K}^{ab} &= \Pi^{ab} - \frac{1}{4} T \epsilon^{abcde} \partial_c A_{de}, \\ S_\alpha &= E_\alpha^M (P_M - T \hat{C}_M) \end{aligned} \quad (47)$$

with

$$\mathcal{P}_a = E_a^M P_M + T (V^a \partial_a Z^M E_M^b \eta_{ba} - \hat{C}_a). \quad (48)$$

The  $\kappa$ -symmetry of the M5-brane action is now reflected (for backgrounds allowing  $\kappa$ -symmetry) in the fact that the fermionic constraints  $\mathcal{S}$  are half first-class and half second-class.

#### 4. Dynamical tension and the null limit

We turn now to the issue of dynamical M5-brane tension. This can be achieved in the Hamiltonian formulation by declaring  $T$  to be an independent variable and then adding to the Lagrangian (46) the new term

$$L' = \dot{\phi}T - u^a \partial_a T, \quad (49)$$

where  $\phi$  is a variable canonically conjugate to  $T$ , and  $u^a$  is a Lagrange multiplier for a new (first-class) constraint. This is the phase space form of the action for a 5-form gauge potential. In fact, eliminating momenta to return to the Lagrangian form we find an equivalent Lagrangian for the original world-volume fields  $(Z^M, A)$  together with a new 5-form gauge potential  $A^{(5)}$ . This Lagrangian can be shown to be the restriction to  $a = t$  of a new six-dimensional Lorentz covariant Lagrangian that depends additionally on the PST field  $a$ . This covariant Lagrangian is

$$L = \frac{1}{v} [L_{\text{DBI}}^2 - (\star G)^2], \quad (50)$$

where  $L_{\text{DBI}}$  is the Lagrangian given in (12) and  $G$  is the ‘modified’ 6-form field strength

$$G = dA^{(5)} - L_{\text{WZ}} - \tilde{H}H. \quad (51)$$

The  $\tilde{H}H$  term is the same as the one in (13); written in differential form notation it is

$$\tilde{H}H \equiv \frac{1}{4(\partial a)^2} da \wedge H \wedge i_{\partial a} H. \quad (52)$$

The reason that this term appears in  $G$  is that the original M5-brane Lagrangian changes by a total derivative under the transformation

$$\delta A = da \wedge \varphi^1(\sigma) \quad (53)$$

with local parameter  $\varphi^1$  [6] (note that  $L_{\text{DBI}}$  is invariant under this transformation). The  $\tilde{H}H$  term therefore behaves like a WZ term with respect to this transformation and its non-invariance is compensated in  $G$  by an appropriate transformation of  $A^{(5)}$ . The Lagrangian (50) is the M5-brane analogue of the super D-brane Lagrangian of [13].

Since we have now reintroduced the PST field we expect the Lagrangian (50) to be invariant under a gauge transformation that will allow the PST field  $a$  to be eliminated by a choice of gauge (e.g. the ‘temporal’ gauge  $a = t$ ). This is not guaranteed by the construction (so far we know only that we recover a Lagrangian equivalent to the original on setting  $a = t$ ) so it must be checked. We expect this gauge invariance to take the form

$$\delta a = \varphi(\sigma), \quad \delta A = \varphi(\sigma) \mathcal{L}^2, \quad (54)$$

together with some variation  $\delta v$  of the Lagrange multiplier  $v$ ; the precise form of  $\mathcal{L}^2$  can be found in [6]. To find the variation of  $v$ , consider a general variation of (50). This has the form

$$\begin{aligned}
\delta L &= \frac{2}{v} [L_{\text{DBI}} \delta L_{\text{DBI}} + \star G \delta(\tilde{H}H + L_{\text{WZ}}) - \star G d(\delta A^5)] - \frac{1}{v^2} [(L_{\text{DBI}})^2 - (\star G)^2] \delta v \\
&= \frac{2}{v} \{ (L_{\text{DBI}} - \star G) \delta L_{\text{DBI}} + \star G [\delta(L_{\text{M5}}) - d\delta A^5] \} \\
&\quad - \frac{1}{v^2} (L_{\text{DBI}} - \star G)(L_{\text{DBI}} + \star G) \delta v,
\end{aligned} \tag{55}$$

where  $L_{\text{M5}} = L_{\text{DBI}} + \tilde{H}H + L_{\text{WZ}}$  is the original Lagrangian of [6]. We know, for *any* of the symmetry transformations of this action that  $\delta L_{\text{M5}} = dA$  for some function  $A$ . This total derivative can be cancelled by a choice of  $\delta A^{(5)}$ . We are therefore left with

$$\delta L = \frac{2}{v} (L_{\text{DBI}} - \star G) \delta L_{\text{DBI}} - \frac{1}{v^2} (L_{\text{DBI}} - \star G)(L_{\text{DBI}} + \star G) \delta v,$$

which vanishes if

$$\delta v = \frac{2v \delta L_{\text{DBI}}}{(L_{\text{DBI}} + \star G)}. \tag{56}$$

It would seem from this result that we have now found a covariant M5-brane action in which the tension is replaced by a 5-form gauge potential, as has been achieved for most other branes. There is a difficulty, however. The denominator of the expression (56) for  $\delta v$  can vanish on the mass-shell. The mass-shell constraint (imposed by the Lagrange multiplier  $v$ ) is

$$0 = (L_{\text{DBI}})^2 - (\star G)^2 = (L_{\text{DBI}} - \star G)(L_{\text{DBI}} + \star G), \tag{57}$$

which implies that  $\star G = \pm L_{\text{DBI}}$ . If we choose the minus sign then the variation  $\delta v$  is singular. Note that this problem occurs *only* for the PST gauge transformation. For  $\kappa$ -transformations, the denominator in (56) is cancelled by a factor appearing in  $\delta_\kappa L_{\text{DBI}}$  (as occurs for super D-branes [13]). For rigid symmetries associated with isometries of the background we have  $\delta L_{\text{DBI}} = 0$  and hence  $\delta v = 0$ . There is therefore no problem for D-branes or any other brane for which a Lorentz covariant action is possible without the introduction of a PST gauge scalar, but there is a problem for the M5-brane. If we demand that the PST gauge variation of  $v$  be non-singular then we are required to choose the solution

$$\star G = L_{\text{DBI}}$$

of the mass-shell constraint. This choice corresponds to the self-duality equations produced by the Lagrangian (11). We conclude that the Lagrangian (50) is classically equivalent to the original M5-brane Lagrangian (11) only if the solutions of the equations of motion of the former are restricted in the way just described. Thus, we have found a version of the covariant M5-brane action of [6] in which the tension appears as a dynamical variable, but it is not completely satisfactory in that it is necessary to supplement the equations of motion that follow from the action with additional information. This is equally a defect of the proposal of [18] for dynamical M5-brane tension.

The unsatisfactory feature of the action (50) that we have just explained has implications for the null, i.e. tensionless, limit. If we take this limit in (50), by setting

$G = 0$ , then the variation (56) becomes singular on the mass shell since the mass-shell constraint is now  $L_{\text{DBI}} = 0$ . This indicates that the zero tension limit can be taken consistently only if one simultaneously sets to zero the world-volume 2-form field  $A$ , because in this case the gauge transformations (54) disappear. In this case we recover the standard null super-5-brane Lagrangian (i.e. without the BI or WZ terms). This is effectively the  $T = 0$  case of (46) because when  $T = 0$  the constraint (38) implies that  $H = 0$  and the conjugate 2-form  $A$  then drops out of the Lagrangian.

Actually, some of these features of the action (50) are already present in the Lagrangian (14). Under (54) the variation of the Lagrange multiplier  $v$  should take the form

$$\delta v = [T - v^{-1} L_{\text{DBI}}]^{-1} [\delta(\tilde{H}H + L_{\text{WZ}}) - d\Lambda], \quad (58)$$

where  $d\Lambda$  is the variation of the original Lagrangian  $L_{\text{M5}}$ . The requirement that this variation be non-singular now singles out one of two possible solutions for  $v$  (namely  $v = -TL_{\text{DBI}}$ ) and the  $T \rightarrow 0$  limit can be taken only if  $H$  is set to zero. In the passage to the Hamiltonian formulation we effectively made this choice by the redefinition (23) of the Lagrange multiplier, which explains why  $H$  drops out of the phase-space action when  $T = 0$ . The problem in the Lagrangian formulation is not solved by gauge fixing the PST invariance because after the gauge fixing the Lagrangian is still invariant (up to a total derivative) under a combination of a PST gauge transformation with a (now non-manifest) world-volume diffeomorphism that preserves the gauge condition. This is an essential symmetry of the non-covariant formulation of the 5-brane action of [8]. Moreover, these non-manifest world-volume diffeomorphisms will be preserved when rewriting the 5-brane Lagrangian in any other form; it is this fact that leads to the transformation properties of the Lagrange multiplier discussed above.

The conclusion seems to be that if we consider the PST gauge invariance and/or (non-manifest) covariance of the 5-brane actions as essential properties then the zero-tension limit leads to an ordinary null super-5-brane, and so does not commute with double dimensional reduction of the 5-brane to a dual IIA D4-brane. If we sacrifice these symmetries and pass to an intrinsically non-covariant formulation of the M5-brane, then the problem with the variation of Lagrange multipliers does not arise. In this case the zero tension limit may commute with the double dimensional reduction, although probably only for those cases that correspond to gauge fixing  $a(\sigma)$  to be the spatial coordinate of the compactified dimension of the 5-brane. Note also that in this case the 5-brane action will yield equations of motion for either a self-dual or an anti-self-dual world-volume field, depending on which solution of (57) is chosen.

## 5. Discussion

We have presented in (46)–(48) a Hamiltonian form of the M5-brane Lagrangian. This result contains various subcases of special interest. For example, the bosonic Lagrangian in a vacuum background is

$$L_{bos} = P \cdot \dot{X} + \frac{1}{2} \Pi^{ab} \dot{A}_{ab} - \lambda \mathcal{H} - s^a \mathcal{H}_a + \sigma_{ab} \mathcal{K}^{ab}, \quad (59)$$

where

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \left[ (P + TV^a \partial_a X)^2 + T^2 \det^5(g + \tilde{H}) \right], \\ \mathcal{H}_a &= \partial_a X \cdot P + TV_a, \\ \mathcal{K}^{ab} &= \Pi^{ab} - \frac{1}{4} T \varepsilon^{abcde} \partial_c A_{de}. \end{aligned} \quad (60)$$

This is the result advertised in [12] (with  $T = 1$  and different factors arising from slightly different conventions).

Another special case is the null M5-brane Lagrangian. Setting  $T = 0$  we find the Lagrangian

$$L|_{T=0} = \dot{Z}^M P_M - \lambda \mathcal{H} - s^a \mathcal{H}_a - \zeta^\alpha \mathcal{S}_\alpha, \quad (61)$$

where

$$\mathcal{H} = \frac{1}{2} \eta^{ab} E_a^M E_b^N P_N P_M, \quad \mathcal{H}_a = \partial_a Z^M P_M, \quad \mathcal{S}_\alpha = E_\alpha^M P_M. \quad (62)$$

A curious feature of the null limit of the M5-brane is that the 2-form gauge potential disappears. As we have seen, this can be traced to the requirement that the tensionless limit be consistent with the PST gauge invariance. This feature, and the related inadequacies of the M5-brane action with dynamical tension, may well be a reflection of the absence of a compelling physical argument in favour of the promotion of the M5-brane tension to a dynamical variable. The basic reason that one needs to elevate a brane tension to the status of a dynamical variable is to accommodate the possibility of it ending on another brane, but the M5-brane cannot have a boundary on another brane. This is the argument against dynamical M5-brane tension given in [17].

This argument can be restated in terms of world-volume domain walls. These can occur as solutions to the brane equations of motion only if discontinuities are allowed in the tension [23], which is possible only if the tension is replaced by a  $p$ -form gauge potential. Consider now the NS5A and NS5B branes. The world-volume solitons on these branes can be interpreted as “little d-branes”. However, in this approach the little d4-brane, or domain wall, is a iib solution of the NS5B brane equations. The systematics is as follows [24]:

- d0-brane: iib (needs  $c^{(1)}$ ) ;
- d1-brane: iia (needs  $c^{(2)+}$ ) ;
- d2-brane: iib (needs  $c^{(3)} = \text{dual of } c^{(1)}$ ) ;
- d3-brane: iia (needs  $c^{(4)} = \text{dual of } c^{(0)}$ ) ;
- d4-brane: iib (needs  $c^{(5)} = \text{dual of } T$ ) ;

where  $c^{(r)}$  is a D-brane world-volume  $r$ -form gauge potential. Therefore one should replace the tension  $T$  of the NS5B brane by a dynamical variable but there is no need to do so for the tension of the NS5A brane. Since the M-theory origin of the NS5A brane is the M5-brane, it follows that neither is there any need to replace the M5-brane

tension by a dynamical variable. Alternatively, one may note that the intersection of an M5-brane with any other M-brane is always over a 1-brane, 3-brane or 5-brane but never over a 4-brane. Therefore, there is never a need to construct a domain wall solution to the M5-brane equations of motion and, correspondingly, there is no need to replace the M5-brane tension by a dynamical variable.

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